

ABSOLUTELY SUMMING OPERATORS AND ATOMIC DECOMPOSITION IN BI-PARAMETER HARDY SPACES.

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Abstract. For $f \in H^p(\delta^2)$, $0 < p \leq 2$, with Haar expansion $f = \sum f_{I \times J} h_{I \times J}$ we constructively determine the Pietsch measure of the 2-summing multiplication operator

$$\mathcal{M}_f : \ell^\infty \rightarrow H^p(\delta^2), \quad (\varphi_{I \times J}) \mapsto \sum \varphi_{I \times J} f_{I \times J} h_{I \times J}.$$

Our method yields a constructive proof of Pisier's decomposition of $f \in H^p(\delta^2)$

$$|f| = |x|^{1-\theta} |y|^\theta \quad \text{and} \quad \|x\|_{X_0}^{1-\theta} \|y\|_{H^2(\delta^2)}^\theta \leq C \|f\|_{H^p(\delta^2)},$$

where X_0 is Pisier's extrapolation lattice associated to $H^p(\delta^2)$ and $H^2(\delta^2)$. Our construction of the Pietsch measure for the multiplication operator \mathcal{M}_f involves the Haar coefficients of f and its atomic decomposition. We treated the one-parameter H^p -spaces in [Houston Journal Math. 2015].

1. INTRODUCTION

Let Y_0, Y be Banach spaces. An operator $T \in L(Y_0, Y)$ is called *2-summing* if there is a constant C such that for every choice of finite sequences (φ_i) in Y_0 , we have

$$(1.1) \quad \left(\sum_{i=1}^n \|T\varphi_i\|^2 \right)^{\frac{1}{2}} \leq C \sup \left\{ \left(\sum_{i=1}^n |\varphi^*(\varphi_i)|^2 \right)^{\frac{1}{2}} : \varphi^* \in B_{Y_0^*} \leq 1 \right\}.$$

In the early 70's the concepts of type and cotype were mainly developed by J. Hoffmann-Jørgensen, S. Kwapien, B. Maurey and G. Pisier, see [HJ74, Kwa72, Mau72a, Mau72b, MP73, MP76, Pis73]. A Banach space Y is called of *cotype 2* if there is a constant C such that for all finite sequences (y_i) in Y

$$\left(\sum_{i=1}^n \|y_i\|^2 \right)^{\frac{1}{2}} \leq C \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) y_i \right\|_X^2 dt \right)^{\frac{1}{2}},$$

where $(r_i)_{i \in \mathbb{N}}$ denotes the independent Rademacher system. One famous theorem due to Maurey ([Mau73, Mau74], see also [Pis78]) combining absolutely summing operators and the concept of cotype states that every bounded operator

$$T : \ell^\infty \rightarrow Y$$

is 2-summing, whenever Y is of cotype 2. In particular, if

$$\|T\varphi\|_Y \leq \sup_{i \in \mathbb{N}} |\varphi_i|,$$

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then T satisfies (1.1) and by Pietsch's factorization theorem (cf. [Woj91]) there exists a constant C such that

$$(1.2) \quad \|T\varphi\|_Y \leq C \left(\int_{\Omega} |\varphi|^2 d\mu \right)^{\frac{1}{2}},$$

where μ is a Borel probability measure on $\Omega = B_{(\ell^\infty)^*}$, called Pietsch measure. Another concept going back to the 70's are Hardy spaces of martingales and their atomic decomposition, cf. [FS72, Fef72, Ber79, Bro80, Gun80, CF80].

In our recent paper [MP15] we exhibited a connection between these two concepts. In the present work we further extend and exploit these newly found connections. We consider operators from ℓ^∞ into bi-parameter dyadic Hardy spaces $H^p(\delta^2)$ that act as multipliers on the Haar system. By the above, these multiplication operators are 2-summing and satisfy therefore (1.2). In our main result (Theorem 3.1) we determine explicit formulae for the Pietsch measure of these multiplication operators. We recall that for general absolutely summing operators the existence of a Pietsch measure is given by a Hahn-Banach argument and is therefore not constructive. Let \mathcal{D} be the set of dyadic intervals. Let $(f_{I \times J})_{I \times J \in \mathcal{D} \times \mathcal{D}}$ be a real sequence indexed by the dyadic rectangles $\mathcal{D} \times \mathcal{D}$. The space $H^p(\delta^2)$ consists of all functions

$$f = \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{D}} f_{I \times J} h_{I \times J},$$

where $h_{I \times J} = h_I \otimes h_J$, which satisfy

$$\|f\|_{H^p(\delta^2)} = \left(\int_{[0,1]^2} \left(\sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{D}} f_{I \times J}^2 1_{I \times J} \right)^{\frac{p}{2}} dm \right)^{\frac{1}{p}} < \infty,$$

where m denotes the Lebesgue measure on $[0,1]^2$. Every $f \in H^p(\delta^2)$ defines a multiplication operator of the form

$$(1.3) \quad \begin{aligned} \mathcal{M}_f: \ell^\infty(\mathcal{D} \times \mathcal{D}) &\rightarrow H^p(\delta^2) \\ (\varphi_{I \times J}) &\mapsto \sum_I \sum_J \varphi_{I \times J} f_{I \times J} h_{I \times J}. \end{aligned}$$

For $1 \leq p \leq 2$ the Hardy spaces $H^p(\delta^2)$ are of cotype 2 and therefore the multiplication operators \mathcal{M}_f are 2-summing and have Pietsch measures. In our main theorem (Theorem 3.1) we use the atomic decomposition of $f \in H^p(\delta^2)$ to give explicit formulae for these Pietsch measures. In particular, we determine $\omega = (\omega_{I \times J})_{I \times J \in \mathcal{D} \times \mathcal{D}}$ with $\omega_{I \times J} \geq 0$ and $\sum \omega_{I \times J} \leq 1$ such that for all $\varphi \in \ell^\infty(\mathcal{D} \times \mathcal{D})$ the following holds

$$(1.4) \quad \|\mathcal{M}_f(\varphi)\|_{H^p(\delta^2)} \leq C \|f\|_{H^p(\delta^2)} \left(\sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{D}} |\varphi_{I \times J}|^2 \omega_{I \times J} \right)^{\frac{1}{2}}.$$

The explicit formulae for ω are given by equation (3.1) in Section 3. Multiplication operators such as given in (1.3) played an important role in the development of Banach space theory. See for instance the proof by Lindenstrauss and Pełczyński on the uniqueness of the unconditional basis in ℓ^1 ([LT77, LP68]).

Bi-parameter Hardy spaces $H^p(\delta^2)$ may be regarded as vector-valued Hardy spaces H_X^p , where $X = H^p$. In [MP15, Theorem 3.3, (3.19)] we obtained partially constructive formulae for the Pietsch measures of Haar multipliers on ℓ^∞ into the vector-valued Hardy spaces H_X^p . In the scalar-valued case, i.e. $X = \mathbb{R}$, we obtained fully constructive formulae for the Pietsch measures of the multiplication operators,

see [MP15, Theorem 3.1.]. With this in mind, our present theorem (Theorem 3.1) gives fully constructive results for a special class of vector-valued Hardy spaces and simultaneously we extend in a non-trivial way the scalar-valued one-parameter case to the bi-parameter case.

Application. The Banach spaces $H^p(\delta^2)$ form Banach lattices whose lattice structure is induced by their unconditional basis $(h_{I \times J})$ and they are related through Calderón's product formula

$$(1.5) \quad H^p(\delta^2) = (H^1(\delta^2))^{1-\theta} (H^2(\delta^2))^\theta, \quad 0 < \theta < 1, \quad \frac{1}{p} = 1 - \theta + \frac{\theta}{2}.$$

This follows by combining the one-parameter identities (cf. [FJ90, Theorem 8.2.]) with Calderón's theorem ([Cal64, Paragraph 13.6]). Therefore, Pisier's extrapolation statement ([Pis79, Theorem 2.10]) can be adapted to the family of $H^p(\delta^2)$ spaces and reads in this setting as follows

$$H^p(\delta^2) = (X_0)^{1-\theta} (H^2(\delta^2))^\theta, \quad \theta = 2 - \frac{2}{p}.$$

Here X_0 is the Banach lattice of all elements $x = \sum_I \sum_J x_{I \times J} h_{I \times J}$ for which

$$(1.6) \quad \|x\|_{X_0} = \sup \left\{ \left\| \sum_I \sum_J |x_{I \times J}|^{1-\theta} |y_{I \times J}|^\theta h_{I \times J} \right\|_{H^p(\delta^2)} \right\} < \infty,$$

where the supremum is taken over all $y = \sum_{I,J} y_{I \times J} h_{I \times J}$ with $\|y\|_{H^2(\delta^2)} \leq 1$. Specifically, (1) asserts that given $f \in H^p(\delta^2)$ there is $x \in X_0$ and $y \in H^2(\delta^2)$ such that

$$(1.7) \quad |f| = |x|^{1-\theta} |y|^\theta \quad \text{and} \quad \|x\|_{X_0}^{1-\theta} \|y\|_{H^2(\delta^2)}^\theta \leq C \|f\|_{H^p(\delta^2)}.$$

Pisier shows in his proof that the weight $\omega = (\omega_{I \times J})$ given by equation (1.4) yields factors for f . Hence, our explicit formulae for $\omega = (\omega_{I \times J})$ determined in Theorem 3.1 allow us to give constructive factors of f satisfying (1.7).

2. PRELIMINARIES

2.1. Bi-parameter Hardy spaces $H^p(\delta^2)$. The *dyadic intervals* \mathcal{D} on the unit interval are given by

$$\mathcal{D} = \{ [2^{-n}(k-1), 2^{-n}k[: n, k \in \mathbb{N}_0, 0 \leq k < 2^n \}$$

and the *dyadic rectangles* \mathcal{R} on the unit square are given by $\mathcal{R} = \mathcal{D} \times \mathcal{D}$. Let $\mathcal{C} \subseteq \mathcal{R}$ be a collection of dyadic rectangles. Then we denote by \mathcal{C}^* the *pointset* covered by the union of all dyadic rectangles in the collection \mathcal{C} . The space $\ell^\infty(\mathcal{R})$ is the space of all sequences $\varphi = (\varphi_{IJ})_{I \times J \in \mathcal{R}}$, indexed by the dyadic rectangles, with $\|\varphi\|_\infty = \sup_{I \times J \in \mathcal{R}} |\varphi_{IJ}| < \infty$. For every $I \in \mathcal{D}$ we define the L^∞ -normalised *Haar function* h_I to be +1 on the left half of I , -1 on the right half of I and zero on $[0, 1] \setminus I$. The an-isotropic *2D Haar system* $(h_{I \times J})_{I \times J \in \mathcal{R}}$ indexed by the dyadic rectangles is defined as follows

$$h_{I \times J}(s, t) := h_I(s) h_J(t), \quad I, J \in \mathcal{D}, (s, t) \in [0, 1]^2.$$

Let $(f_{IJ})_{I \times J \in \mathcal{R}}$ be a real sequence and $f = (f_{IJ})_{I \times J \in \mathcal{R}}$ the real vector indexed by the dyadic rectangles. The *square function* of f is defined as follows

$$S(f)(s, t) = \left(\sum_{I \times J \in \mathcal{R}} f_{IJ}^2 1_{I \times J}(s, t) \right)^{\frac{1}{2}}, \quad (s, t) \in [0, 1]^2.$$

The *bi-parameter dyadic Hardy space* $H^p(\delta^2)$, $0 < p \leq 2$, consists of vectors $f = (f_{IJ})_{I \times J \in \mathcal{R}}$ for which

$$\|f\|_{H^p(\delta^2)} = \left(\int_{[0,1]^2} S^p(f)(s, t) dm(s, t) \right)^{\frac{1}{p}} < \infty,$$

where m is the Lebesgue measure on $[0, 1]^2$. Systematically we use the notation $\|f\|_2 = \|f\|_{H^2(\delta^2)}$. For convenience we identify $f = (f_{IJ})_{I \times J \in \mathcal{R}} \in H^p(\delta^2)$ with its formal Haar series

$$(2.1) \quad f = \sum_{I \times J \in \mathcal{R}} f_{IJ} h_{I \times J}.$$

2.2. Atomic decomposition. Let $0 < p \leq 2$ and $f \in H^p(\delta^2)$ with Haar expansion (2.1). For every $n \in \mathbb{Z}$ we define the set

$$F_n = \{(s, t) \in [0, 1]^2 : S(f)(s, t) > 2^n\}$$

and the collection of dyadic rectangles

$$\mathcal{R}_n = \left\{ I \times J \in \mathcal{R} : |I \times J \cap F_n| > \frac{|I \times J|}{2}, |I \times J \cap F_{n+1}| \leq \frac{|I \times J|}{2} \right\}.$$

Then $f = \sum_{n \in \mathbb{Z}} f_n$, where

$$f_n = \sum_{I \times J \in \mathcal{R}_n} f_{IJ} h_{I \times J}$$

and the following inequalities hold

$$(2.2) \quad \|f\|_{H^p(\delta^2)}^p \leq \sum_{n \in \mathbb{Z}} \|f_n\|_{H^p(\delta^2)}^p \leq \sum_{n \in \mathbb{Z}} |\mathcal{R}_n^*|^{1-\frac{p}{2}} \|f_n\|_2^p \leq A_p \|f\|_{H^p(\delta^2)}^p.$$

The family $(f_n, \mathcal{R}_n)_{n \in \mathbb{Z}}$ is called the *atomic decomposition* of $u \in H^p(\delta^2)$. This decomposition originates from [Fef72, Ber79, Gun80, CF80].

Note that the right-hand side inequality in (2.2) results from the following

$$(2.3) \quad \begin{aligned} \|f_n\|_2^2 &= \int_{[0,1]^2} S^2(f_n) dm \leq 2 \int_{[0,1]^2} S^2(f_n) 1_{F_{n+1}^c} dm \leq 2 \cdot 2^{2(n+1)} |\mathcal{R}_n^*| \\ &\leq 8 \cdot 2^{2n} \left| \left\{ M_S(1_{F_n}) > \frac{1}{2} \right\} \right| \leq C 2^{2n} |F_n|. \end{aligned}$$

Here M_S is the strong maximal operator (cf. [JMZ35], [FFW95]) in $[0, 1]^2$ given by

$$M_S(1_{F_n})(s, t) = \sup_{R \ni (s, t)} \frac{1}{|R|} \int_R 1_{F_n} dm,$$

where the supremum is taken over all rectangles R in $[0, 1]^2$ with side length parallel to the axes. Boundedness estimates for the strong maximal operator (cf. [JMZ35]) give rise to bi-parameter Fefferman-Stein strong maximal operator estimates (cf. [FS71, Theorem 1.]). We exploit these Fefferman-Stein inequalities in the following form.

Lemma 2.1. Fix $\varepsilon > 0$. Suppose that for each $I \times J \in \mathcal{R}$ the subset $E_{I \times J} \subseteq I \times J$ is a measurable set with $\frac{|E_{I \times J}|}{|I \times J|} > \varepsilon$. Then for any $f \in H^p(\delta^2)$, $0 < p < \infty$, with Haar expansion $f = \sum_{I \times J \in \mathcal{R}} f_{IJ} h_{I \times J}$, the following holds

$$\|f\|_{H^p(\delta^2)} \leq C_p(\varepsilon) \left\| \left(\sum_{I \times J \in \mathcal{R}} |f_{IJ}|^2 1_{E_{I \times J}} \right)^{\frac{1}{2}} \right\|_{L^p}.$$

Frazier and Jawerth ([FJ90, Theorem 2.7.]) give a proof for the one-parameter version of this lemma. Their proof can be adapted to the setting above.

2.3. Modified Hölder inequality. See [HLP52, p. 61 (65.)]. Let (Ω, Σ, μ) be a measure space and $r > 1$ or $r < 0$. Then for all measurable functions f, g on Ω

$$(2.4) \quad \int_{\Omega} f^r g^{1-r} d\mu \geq \left(\int_{\Omega} f d\mu \right)^r \left(\int_{\Omega} g d\mu \right)^{1-r}.$$

3. THE MAIN THEOREM

Let $0 < p \leq 2$. Every $f \in H^p(\delta^2)$ defines a multiplication operator of the form

$$\mathcal{M}_f: \ell^\infty(\mathcal{R}) \rightarrow H^p(\delta^2), (\varphi_{IJ}) \mapsto \sum_{I \times J \in \mathcal{R}} \varphi_{IJ} f_{IJ} h_{I \times J}$$

and clearly we have

$$\|\mathcal{M}_f: \ell^\infty(\mathcal{R}) \rightarrow H^p(\delta^2)\| \leq \|f\|_{H^p(\delta^2)}.$$

Banach space theory as described in the introduction guarantees that these multiplication operators are 2-summing and satisfy (1.4). In Theorem 3.1 we determine explicit formulae for the weights $\omega = (\omega_{I \times J})$ given in (1.4). Every multiplication operator \mathcal{M}_f is induced by a function $f \in H^p(\delta^2)$. These functions admit an atomic decomposition $(f_n, \mathcal{R}_n)_{n \in \mathbb{N}}$ satisfying the equations in (2.2). This is the input for our construction and the output is equation (3.1) determining ω explicitly.

Theorem 3.1. Let $0 < p \leq 2$ and $f \in H^p(\delta^2)$ with Haar expansion

$$f = \sum_{I \times J \in \mathcal{R}} f_{IJ} h_{I \times J}$$

and atomic decomposition $(f_n, \mathcal{R}_n)_{n \in \mathbb{Z}}$. Then the sequence $(\omega_{IJ})_{I \times J \in \mathcal{R}}$, defined by

$$(3.1) \quad \omega_{IJ} = \frac{1}{A_p \|f\|_{H^p(\delta^2)}^p} \frac{|\mathcal{R}_n^*|^{1-\frac{p}{2}} f_{IJ}^2 |I||J|}{\|f_n\|_2^{2-p}}, \quad I \times J \in \mathcal{R}_n,$$

satisfies

$$\sum_{I \times J \in \mathcal{R}} \omega_{IJ} \leq 1$$

and for each $\varphi \in \ell^\infty(\mathcal{R})$ the following inequality holds

$$\|\mathcal{M}_f(\varphi)\|_{H^p(\delta^2)} \leq A_p \|f\|_{H^p(\delta^2)} \left(\sum_{I \times J \in \mathcal{R}} |\varphi_{IJ}|^2 \omega_{IJ} \right)^{\frac{1}{2}}.$$

Proof. Note that the left-hand side inequality of (2.2) depends only on the fact that $(\mathcal{R}_n)_{n \in \mathbb{Z}}$ forms a partition of \mathcal{R} . Hence, for all $\varphi = (\varphi_{IJ}) \in \ell^\infty(\mathcal{R})$ the following estimate holds

$$\begin{aligned} \left\| \sum_{I \times J \in \mathcal{R}} \varphi_{IJ} f_{IJ} h_{I \times J} \right\|_{H^p(\delta^2)}^p &= \left\| \sum_{n \in \mathbb{Z}} \sum_{I \times J \in \mathcal{R}_n} \varphi_{IJ} f_{IJ} h_{I \times J} \right\|_{H^p(\delta^2)}^p \\ &\leq \sum_{n \in \mathbb{Z}} \left\| \sum_{I \times J \in \mathcal{R}_n} \varphi_{IJ} f_{IJ} h_{I \times J} \right\|_2^p |\mathcal{R}_n^*|^{1-\frac{p}{2}} \\ &= \sum_{n \in \mathbb{Z}} \left\| \sum_{I \times J \in \mathcal{R}_n} \varphi_{IJ} \frac{f_{IJ}}{\|f_n\|_2} h_{I \times J} \right\|_2^p \|f_n\|_2^p |\mathcal{R}_n^*|^{1-\frac{p}{2}}. \end{aligned}$$

With

$$\left\| \sum_{I \times J \in \mathcal{R}_n} \varphi_{IJ} \frac{f_{IJ}}{\|f_n\|_2} h_{I \times J} \right\|_2^p = \left(\sum_{I \times J \in \mathcal{R}_n} \varphi_{IJ}^2 \frac{f_{IJ}^2}{\|f_n\|_2^2} |I||J| \right)^{\frac{p}{2}}$$

and Hölder's inequality we get

$$\begin{aligned} \left\| \sum_{I \times J \in \mathcal{R}} \varphi_{IJ} f_{IJ} h_{I \times J} \right\|_{H^p(\delta^2)}^p &\leq \sum_{n \in \mathbb{Z}} \left(\sum_{I \times J \in \mathcal{R}_n} \varphi_{IJ}^2 \frac{f_{IJ}^2}{\|f_n\|_2^2} |I||J| \right)^{\frac{p}{2}} \|f_n\|_2^p |\mathcal{R}_n^*|^{1-\frac{p}{2}} \\ &= \sum_{n \in \mathbb{Z}} \left(\sum_{I \times J \in \mathcal{R}_n} \varphi_{IJ}^2 \frac{f_{IJ}^2}{\|f_n\|_2^2} |I||J| \|f_n\|_2^p |\mathcal{R}_n^*|^{1-\frac{p}{2}} \right)^{\frac{p}{2}} \left(\|f_n\|_2^p |\mathcal{R}_n^*|^{1-\frac{p}{2}} \right)^{1-\frac{p}{2}} \\ &\leq \left(\sum_{n \in \mathbb{Z}} \sum_{I \times J \in \mathcal{R}_n} \varphi_{IJ}^2 \frac{f_{IJ}^2}{\|f_n\|_2^{2-p}} |I||J| |\mathcal{R}_n^*|^{1-\frac{p}{2}} \right)^{\frac{p}{2}} \left(\sum_{n \in \mathbb{Z}} \|f_n\|_2^p |\mathcal{R}_n^*|^{1-\frac{p}{2}} \right)^{1-\frac{p}{2}}. \end{aligned}$$

$$\begin{aligned} \left\| \sum_{I \times J \in \mathcal{R}} \varphi_{IJ} f_{IJ} h_{I \times J} \right\|_{H^p(\delta^2)}^p &\leq A_p^{1-\frac{p}{2}} \|f\|_{H^p(\delta^2)}^{p(1-\frac{p}{2})} \left(\sum_{n \in \mathbb{Z}} \sum_{I \times J \in \mathcal{R}_n} \varphi_{IJ}^2 \frac{f_{IJ}^2}{\|f_n\|_2^{2-p}} |I||J| |\mathcal{R}_n^*|^{1-\frac{p}{2}} \right)^{\frac{p}{2}} \\ &= A_p \|f\|_{H^p(\delta^2)}^p \left(\sum_{n \in \mathbb{Z}} \sum_{I \times J \in \mathcal{R}_n} \varphi_{IJ}^2 \frac{f_{IJ}^2}{\|f_n\|_2^{2-p} \|f\|_{H^p(\delta^2)}^p} |I||J| |\mathcal{R}_n^*|^{1-\frac{p}{2}} \right)^{\frac{p}{2}}. \end{aligned}$$

Recall that

$$(3.2) \quad \|f_n\|_2^2 = \sum_{I \times J \in \mathcal{R}_n} f_{IJ}^2 |I||J|.$$

By the right-hand side inequality in equation (2.2) and by equation (3.2) we obtain for the sequence $(\omega_{IJ})_{I \times J \in \mathcal{R}}$, defined by

$$\omega_{IJ} = \frac{1}{A_p \|f\|_{H^p(\delta^2)}^p} \frac{|\mathcal{R}_n^*|^{1-\frac{p}{2}} f_{IJ}^2 |I||J|}{\|f_n\|_2^{2-p}}, \quad I \times J \in \mathcal{R}_n,$$

the following estimate

$$\begin{aligned} \sum_{I \times J \in \mathcal{R}} \omega_{IJ} &= \frac{1}{A_p \|f\|_{H^p(\delta^2)}^p} \sum_{n \in \mathbb{Z}} \sum_{I \times J \in \mathcal{R}_n} \frac{|\mathcal{R}_n^*|^{1-\frac{p}{2}} f_{IJ}^2 |I||J|}{\|f_n\|_2^{2-p}} \\ &= \frac{1}{A_p \|f\|_{H^p(\delta^2)}^p} \sum_{n \in \mathbb{Z}} |\mathcal{R}_n^*|^{1-\frac{p}{2}} \|f_n\|_2^p \leq 1. \end{aligned}$$

□

4. ANOTHER APPLICATION OF THE ATOMIC DECOMPOSITION

Pisier's extrapolation lattice X_0 defined in (1.6) is known to coincide with $H^1(\delta^2)$. This follows by a specialisation of a general theorem of Cwikel and Nilsson (see [CNS03]). Their extrapolation method is applicable, since $H^p(\delta^2)$ spaces are related through Calderon's product formula (cf. equation (1.5)). The space X_0 is of particular importance to our work in this paper. Hence, we take the opportunity to complement the work of [CNS03, FJ90] with a direct argument based on the atomic decomposition of $H^p(\delta^2)$. We build our strategy by exploiting the formulae used by [Mül05, GMP05, Bow13] for similar purposes. In particular, we refer to Bownik's paper [Bow13] for the formula (4.3) and the idea of using Lemma 2.1 in the proof of the following theorem.

Theorem 4.1. *Let $f \in H^p(\delta^2)$, $0 < p \leq 2$, with Haar expansion $f = \sum f_{IJ} h_{I \times J}$. Then for $0 < \theta < 1$ and q given by*

$$(4.1) \quad \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2},$$

the following holds:

$$c_p \|f\|_{H^p(\delta^2)}^{1-\theta} \leq \sup \left\{ \left\| \sum_{I \times J \in \mathcal{R}} |f_{IJ}|^{1-\theta} |g_{IJ}|^\theta h_{I \times J} \right\|_{H^q(\delta^2)} \right\} \leq \|f\|_{H^p(\delta^2)}^{1-\theta},$$

where the supremum is taken over all functions $g = \sum g_{IJ} h_{I \times J}$ with $\|g\|_2 \leq 1$.

Proof. We start with the proof of the right-hand side inequality. Let

$$h = \sum_{I \times J \in \mathcal{R}} |f_{IJ}|^{1-\theta} |g_{IJ}|^\theta h_{I \times J}.$$

Then, by applying Hölder's inequality for sequence spaces with $1 - \theta + \theta = 1$, we obtain the following inequality for the square functions

$$S^q(h) \leq S^{q(1-\theta)}(f) S^{q\theta}(g).$$

Integrating over $[0, 1]^2$ and applying Hölder's inequality with $\frac{q(1-\theta)}{p} + \frac{q\theta}{2} = 1$ yields

$$\|h\|_{H^q(\delta^2)} \leq \|f\|_{H^p(\delta^2)}^{1-\theta} \|g\|_2^\theta.$$

For the left-hand side inequality we show that for every $f \in H^p(\delta^2)$ there exists a function $g \in H^2(\delta^2)$ such that $\|g\|_2^2 \leq c_p \|f\|_{H^p(\delta^2)}^p$ and

$$(4.2) \quad \left\| \sum_{I \times J \in \mathcal{R}} |f_{IJ}|^{1-\theta} |g_{IJ}|^\theta h_{I \times J} \right\|_{H^q(\delta^2)}^q \geq C_p \|f\|_{H^p(\delta^2)}^p.$$

Let $(f_n, \mathcal{R}_n)_{n \in \mathbb{Z}}$ be the atomic decomposition of $f \in H^p(\delta^2)$. Let $g = \sum g_{IJ} h_{I \times J}$, where

$$(4.3) \quad |g_{IJ}| = 2^{-\frac{n}{2}(2-p)} |f_{IJ}|, \quad I \times J \in \mathcal{R}_n.$$

Then, by equation (2.3), we have

$$(4.4) \quad \begin{aligned} \|g\|_2^2 &= \sum_{n \in \mathbb{Z}} 2^{-n(2-p)} \|f_n\|_2^2 \leq C \sum_{n \in \mathbb{Z}} 2^{-n(2-p)} 2^{2n} |F_n| = C \sum_{n \in \mathbb{Z}} 2^{np} |F_n| \\ &\leq c_p \|f\|_{H^p(\delta^2)}^p. \end{aligned}$$

To prove equation (4.2) we use Lemma 2.1 with sets $E_{I \times J} = I \times J \cap F_n$, for $I \times J \in \mathcal{R}_n$ and obtain

$$(4.5) \quad \begin{aligned} \|f\|_{H^p(\delta^2)}^p &= \left(\int_{[0,1]^2} S^p(f) dm \right)^{\frac{q}{p}} \left(\int_{[0,1]^2} S^p(f) dm \right)^{1-\frac{q}{p}} \\ &\leq C_p^q \left(\int_{[0,1]^2} \left(\sum_{I \times J} |f_{IJ}|^2 1_{E_{I \times J}} \right)^{\frac{p}{2}} dm \right)^{\frac{q}{p}} \left(\int_{[0,1]^2} S^p(f) dm \right)^{1-\frac{q}{p}}. \end{aligned}$$

Let $h = \left(\sum_{I \times J \in \mathcal{R}} |f_{IJ}|^2 1_{E_{I \times J}} \right)^{\frac{1}{2}}$. Note that by the modified Hölder inequality (cf. equation (2.4)) we have

$$(4.6) \quad \left(\int_{[0,1]^2} h^p dm \right)^{\frac{q}{p}} \left(\int_{[0,1]^2} S^p(f) dm \right)^{1-\frac{q}{p}} \leq \int_{[0,1]^2} h^q S^{p-q}(f) dm.$$

Combining equation (4.5) and (4.6) yields

$$(4.7) \quad \begin{aligned} \|f\|_{H^p(\delta^2)}^p &\leq C_p^q \int_{[0,1]^2} \left(\sum_{I \times J \in \mathcal{R}} |f_{IJ}|^2 1_{E_{I \times J}} \right)^{\frac{p}{2}} S^{p-q}(f) dm \\ &= C_p^q \int_{[0,1]^2} \left(\sum_{n \in \mathbb{Z}} \sum_{I \times J \in \mathcal{R}_n} |f_{IJ}|^2 1_{I \times J} 1_{F_n} \right)^{\frac{p}{2}} S^{p-q}(f) dm. \end{aligned}$$

We know that $S(f) 1_{F_n} > 2^n 1_{F_n}$. Since $q > p$, it follows that

$$(4.8) \quad S(f)^{p-q} 1_{F_n} < 2^{-n(q-p)} 1_{F_n}.$$

Equation (4.1) gives the identity $q - p = \frac{q\theta}{2}(2 - p)$. Hence, putting equation (4.8) into equation (4.7) yields

$$(4.9) \quad \begin{aligned} \|f\|_{H^p(\delta^2)}^p &\leq C_p^q \int_{[0,1]^2} \left(\sum_{n \in \mathbb{Z}} 2^{-n\theta(2-p)} \sum_{I \times J \in \mathcal{R}_n} |f_{IJ}|^2 1_{I \times J} 1_{F_n} \right)^{\frac{p}{2}} dm \\ &\leq C_p^q \int_{[0,1]^2} \left(\sum_{n \in \mathbb{Z}} 2^{-n\theta(2-p)} \sum_{I \times J \in \mathcal{R}_n} |f_{IJ}|^2 1_{I \times J} \right)^{\frac{p}{2}} dm \\ &= C_p^q \int_{[0,1]^2} \left(\sum_{I \times J \in \mathcal{R}} |f_{IJ}|^{2(1-\theta)} |g_{IJ}|^{2\theta} 1_{I \times J} \right)^{\frac{p}{2}} dm \\ &= C_p^q \left\| \sum_{I \times J \in \mathcal{R}} |f_{IJ}|^{1-\theta} |g_{IJ}|^\theta h_{I \times J} \right\|_{H^q(\delta^2)}^q. \end{aligned}$$

Summarizing equations (4.4) and (4.9) yields

$$\|f\|_{H^p(\delta^2)}^{1-\theta} \|g\|_2^\theta \leq c_p^{\frac{\theta}{2}} \|f\|_{H^p(\delta)}^{\frac{p}{q}} \leq C_p \left\| \sum_{I \times J \in \mathcal{R}} |f_{IJ}|^{1-\theta} |g_{IJ}|^\theta h_{I \times J} \right\|_{H^q(\delta^2)}.$$

□

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